Topological Ramsey theory of countable ordinals

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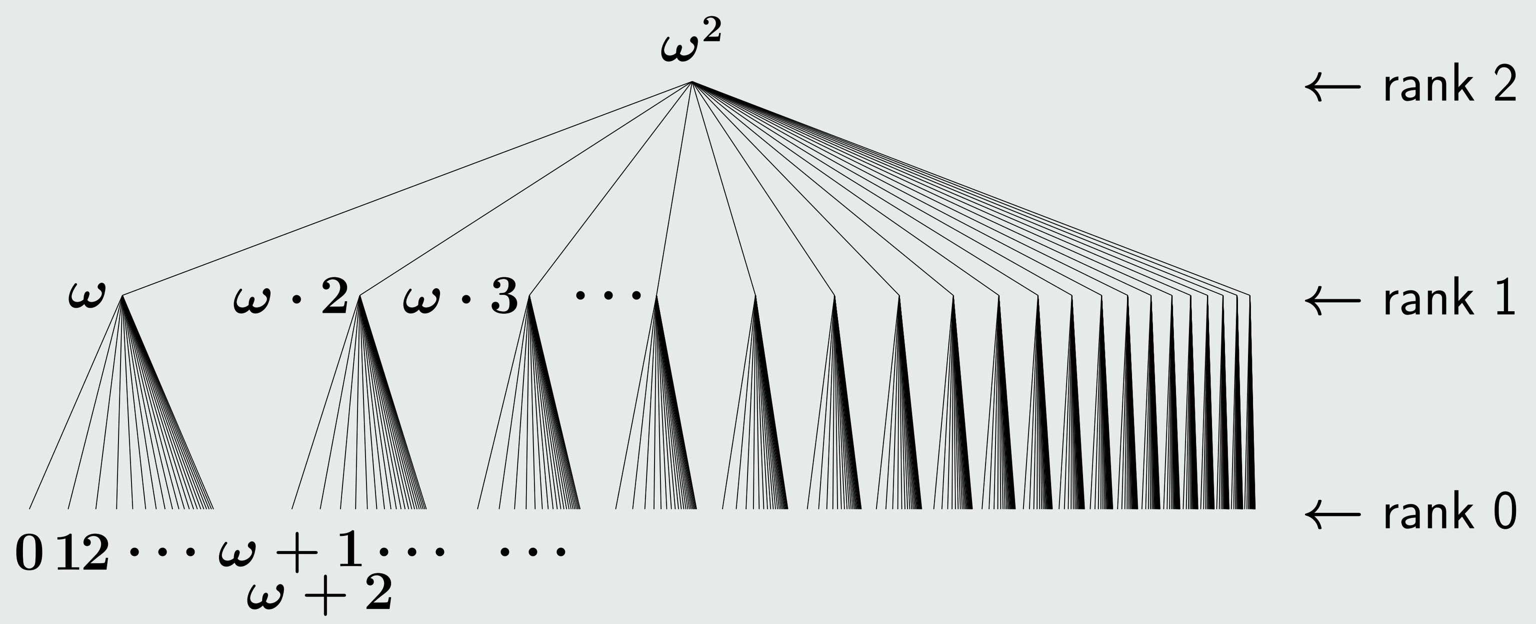
Ordinal topologies

The *order topology* on a totally ordered set X is generated by intervals (x, y) with $x \in X \cup \{-\infty\}$ and $y \in X \cup \{\infty\}$. This generalises the Euclidean topology on $\mathbb{R}$.

When ordinals are endowed with the order topology, points corresponding to non-zero limit ordinals become topological limits of the points below them. For example, $\omega + 1 \cong \{1 - \frac{1}{n} : n \in \mathbb{N}\} \cup \{1\}$.

The Cantor–Bendixson *derivative* X' of a topological space X is the set of limit points of X. The *rank* of $x \in X$ is the number of iterated derivatives that can be applied before x is removed. In general, this rank may be ordinal-valued or may not exist.

We visualise ordinal topologies by positioning a point vertically according to its rank. For example, here is $\omega^2 + 1$:



Topological partition relations

$K_X :=$ complete graph with vertex set X.

The usual Erdős–Rado partition relation $\kappa \to (\lambda)^2_2$ between cardinals κ and λ means that, for every red-blue edge-colouring of K_κ, there is a set of vertices of cardinality λ, between which every edge has the same colour. For example, $\aleph_0 \to (\aleph_0)^2_2$ is Ramsey's theorem.

In a topological partition relation, we replace cardinals with topological spaces.

Definition. Let Y, X_{red} and X_{blue} be topological spaces. We write $Y \to_{top} (X_{\text{red}}, X_{\text{blue}})^2$ to mean that, for every red-blue edge-colouring of K_Y, there is either a complete red subgraph on a subspace homeomorphic to X_{red}, or a complete blue subgraph on a subspace homeomorphic to X_{blue}.

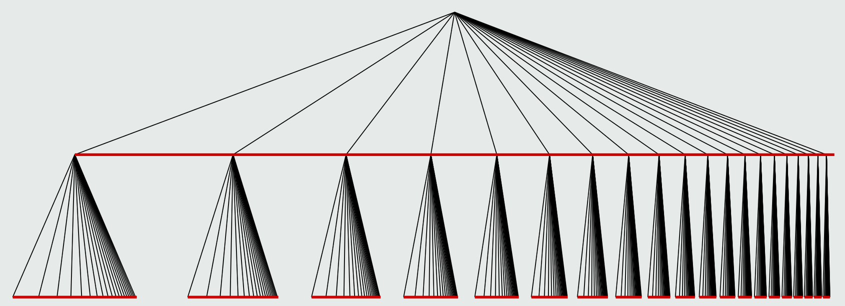
For example, if Y, X_{red} and X_{blue} are discrete spaces, then we recover the usual partition relation. See [5] for further details.

Example

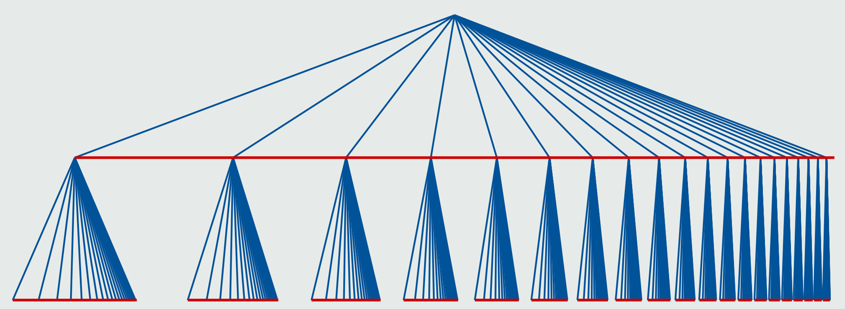
Theorem. $\omega^2 + 1 \to_{top} (\omega + 1, 3)^2$.

Proof. Suppose we are given a red-blue edge-colouring of K_{ω^2+1}. We must find either a complete red subgraph on a subspace homeomorphic to $\omega + 1$, or a blue triangle. Note that a subspace homeomorpic to $\omega + 1$ is simply any subset of order type ω together with its supremum.

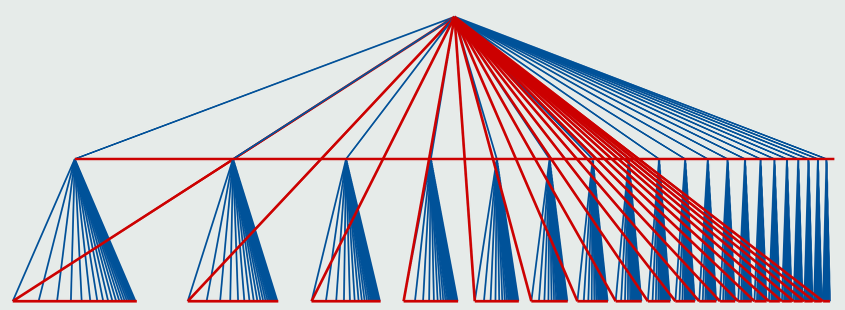
Given any infinite subset of $\omega^2 + 1$, by Ramsey's theorem there is either a blue triangle, or an infinite complete red subgraph. By passing to a subset if necessary, we may therefore assume that these are complete red subgraphs:



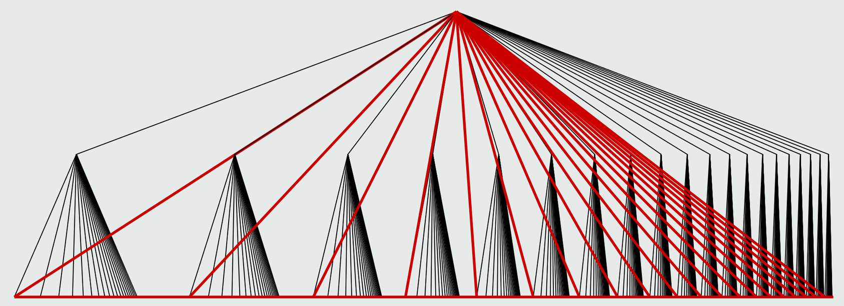
Next, given any infinite set of edges, infinitely many of them have the same colour. By passing again to a subset if necessary, we may assume that these edges are blue, otherwise we obtain a complete red subgraph on a subspace homeomorphic to $\omega + 1$:



To avoid a blue triangle, we may now assume that these edges are red:



Finally, consider only those vertices incident to the red edges we have just identified. By Ramsey's theorem, we may once again assume that there is an infinite complete red subgraph:



This gives us a complete red subgraph on a vertex set homeomorphic to $\omega + 1$, as required. □

Topological Ramsey numbers

Definition. Let α and β be ordinals. The *topological ordinal Ramsey number* $R^{top}(\alpha, \beta)$ is the least ordinal γ (if one exists) such that $\gamma \to_{top} (\alpha, \beta)^2$.

Since $\omega^2 \not\to_{top} (\omega + 1, 3)^2$ (exercise), it follows from our example that $R^{top}(\omega + 1, 3) = \omega^2 + 1$.

If γ is a countable ordinal, then using two orderings and a Sierpiński "same-different" colouring, one can show that $\gamma \not\to_{top} (\omega + 1, \omega)^2$. Hence if $R^{top}(\alpha, \beta)$ is countable and $\alpha > \omega$, then β must be finite.

We therefore studied $R^{top}(\alpha, k)$ with α countable and k finite. Building on work from [2] and [3], we proved these results in [1].

Theorem (Caicedo–Hilton, 2015)**.** Let α be a countable ordinal and let k be a positive integer.

- $R^{top}(\omega + 1, k + 1) = \omega^k + 1$.
- $R^{top}(\alpha, k) < \omega^\omega$ if $\alpha < \omega^2$.
- $R^{top}(\omega^2, k) \leq \omega^\omega$.
- $R^{top}(\omega^2 + 1, k + 2) \leq \omega^{\omega \cdot k} + 1$.
- $R^{top}(\omega^n + 1, k + 2) \leq \omega^{\omega^k \cdot n} + 1$ for every positive integer n.
- $R^{top}(\omega^{\omega^\alpha}, k + 1) \leq \omega^{\omega^{\alpha \cdot k}}$.

In particular, $R^{top}(\alpha, k)$ is indeed countable whenever α is countable and k is finite, by the last of these results. This is a topological version of the Erdős–Milner theorem of [4].

References

[1] Andrés E Caicedo and Jacob Hilton. Topological Ramsey numbers and countable ordinals. *arXiv preprint arXiv:1510.00078*, 2015.

[2] Jacob Hilton. The topological pigeonhole principle for ordinals. *The Journal of Symbolic Logic*, to appear. *arXiv preprint arXiv:1410.2520*, 2014.

[3] James E Baumgartner. Partition relations for countable topological spaces. *Journal of Combinatorial Theory, Series A*, 43(2):178–195, 1986.

[4] Paul Erdős and Eric C Milner. A theorem in the partition calculus. *Canadian Mathematical Bulletin*, 15(4):501, 1972.

[5] William Weiss. Partitioning topological spaces. In *Mathematics of Ramsey theory*, pages 154–171. Springer, 1990.