Topological Ramsey theory of countable ordinals

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Ramsey's theorem

$K_X :=$ complete graph with vertex set X

Theorem (Ramsey, 1930)

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Proof: blackboard.

Ordinal Ramsey numbers

Definition

Let α, β and γ be ordinals.

$$\gamma \to (\alpha, \beta)^2$$

means: given a red-blue edge-colouring of K_γ, there is either a complete red subgraph with vertex set of order type α, or a complete blue subgraph with vertex set of order type β.

$R(\alpha, \beta) :=$ least γ such that $\gamma \to (\alpha, \beta)^2$.

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$R(\omega, \omega) = \omega$ (Ramsey's theorem)

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$R(\omega + 1, k + 1) = \omega \cdot k + 1$ (for all finite k, by induction)

Why study $R(\alpha, k)$ for countable α and finite k?

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Theorem (Specker, 1956)

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Hence if $\alpha > \omega$ and $R(\alpha, \beta)$ is countable, then β must be finite.

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Proof: put two orderings on γ: the usual one, and one of type ω. Colour an edge xy red iff the orderings agree about x and y.

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Theorem (Erdős–Milner, 1972)

If α is countable and k is finite, then $R(\alpha, k)$ is countable.

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We say an ordinal α is *indecomposable* to mean:
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ω^α is indecomposable for all α.

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Proof for countable α: by induction (blackboard).

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They proved: $R(\omega^{1+\alpha}, 2^k) \leq \omega^{1+\alpha \cdot k}$.
We prove: $R(\omega^{\alpha}, k+1) \leq \omega^{\alpha \cdot k}$.

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Let α, β and γ be ordinals.

$$\gamma \rightarrow_{top} (\alpha, \beta)^2$$

means: given a red-blue edge-colouring of K_γ, there is either a complete red subgraph with vertex set **homeomorphic** to α, or a complete blue subgraph with vertex set **homeomorphic** to β (using the order topology on γ).

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For example, a subspace $X \subseteq \gamma$ is homeomorphic to $\omega + 1$ iff X has order type $\omega + 1$ and $\max X = \sup(X \setminus \{\max X\})$.

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$R(\omega^{1+\alpha}, 2^k) \leq \omega^{1+\alpha \cdot k}$ (Erdős–Milner, 1972)	$(R^{top}(\omega^{\omega^\alpha}, 2^k) \leq \omega^{\omega^{\alpha \cdot k}}?)$

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$R(\omega^{1+\alpha}, 2^k) \leq \omega^{1+\alpha \cdot k}$ (Erdős–Milner, 1972)	$(R^{top}(\omega^{\omega^\alpha}, 2^k) \leq \omega^{\omega^{\alpha \cdot k}}?)$
algorithm to compute $R(\alpha, k)$ for all $\alpha < \omega^\omega$ (Haddad–Sabbagh, 1969)	$R^{top}(\alpha, k) < \omega^\omega$ for all $\alpha < \omega^2$ $R^{top}(\omega^2, k) \leq \omega^\omega$ $R^{top}(\omega^2+1, k+2) \leq \omega^{\omega \cdot k} + 1$

Thank you!

Thank you for your attention!

A preprint of our paper is available at
`http://arxiv.org/abs/1510.00078`.